

# Radiation of scalar oscillons in 2 and 3 dimensions

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## Abstract

The radiation loss of small-amplitude radially symmetric oscillons (long-living, spatially localized, time-dependent solutions) in two- and three-dimensional scalar field theories is computed analytically in the small-amplitude expansion. The amplitude of the radiation is beyond all orders in perturbation theory and it is determined using matched asymptotic series expansions and Borel summation. The general results are illustrated on the case of the two- and three-dimensional sine-Gordon theory and a two-dimensional  $\phi^6$  model. The analytic predictions are found to be in good agreement with the results of numerical simulations of oscillons.

## 1. Introduction

There is an increasing interest in long-living, spatially localized classical solutions in field theories – *oscillons* – exhibiting nearly periodic oscillations in time [1]–[16]. For the two and more spatial dimensional sine-Gordon theory, following the original naming in [3], these objects are generally called *pulsions* [17]–[21]. Oscillons resemble the “true” (time periodic and of finite energy) breathers of the one-dimensional sine-Gordon (SG) theory, but unlike true breathers they are continuously losing energy by radiating slowly. Just like a breather, an oscillon possesses a localized “core”, but it also has a “radiative” region outside of the core. Oscillons appear from rather generic initial data in the course of time evolution, in an impressive number of physically relevant theories including the bosonic sector of the standard model [22]–[25]. Moreover they form in physical processes making them of considerable importance [26]–[33].

A crucial physical characteristic of oscillons is the amplitude of the outgoing wave determining their lifetime. In a previous work we have been able to perform an analytic calculation of the radiation amplitude of oscillons in scalar theories in the limit of small oscillon amplitudes in one spatial dimension [34]. In this limit one can perform an expansion yielding breather-like configurations with spatially localized cores. The small-amplitude expansion yields an asymptotic series for the core, but misses a standing wave tail whose amplitude is exponentially small with respect to that of the core [35]. Therefore in order to compute the radiative amplitude, more precisely its leading part, implies to go beyond all orders in perturbation theory. In the present paper building on the results of [34] we develop a method to compute the radiation amplitude in  $D = 2$  and  $D = 3$  spatial dimensions in the small-amplitude expansion. The main result of this paper can be summarized by the following simple formula determining the energy

loss of a small-amplitude oscillon in  $D$  spatial dimensions ( $D < 4$ ):

$$\frac{dE}{dt} = -\frac{A_D}{\varepsilon^{D-1}} \exp\left(-\frac{B_D}{\varepsilon}\right), \quad (1)$$

where  $A_D, B_D$  are given in Eq. (45) as functions of  $D$  and of the coefficients of the scalar potential. Although our analytic calculations are carried out in the limit of the oscillon amplitude going to zero, the results are valid for non infinitesimal values of the amplitude. We have found satisfactory agreement between the predicted energy loss Eq. (1) and the “measured” one (by numerical simulations) for the SG theory in  $D = 2$  and  $D = 3$  and in a  $\phi^6$ -type model investigated in Ref. [34] in  $D = 2$ . Our results also imply that no breathers depending on continuous parameters can exist for  $D > 1$  for arbitrary scalar potentials. Note that in  $D = 1$  the SG theory is the *only* one with analytic potential admitting a breather family [36].

## 2. Small-amplitude expansion

We consider spherically symmetric solutions of a real scalar theory in a  $D + 1$  dimensional Minkowski space-time, with a self-interaction potential  $U(\phi)$ . The equation of motion is

$$-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial r^2} + \frac{D-1}{r} \frac{\partial \phi}{\partial r} = U'(\phi) = \phi + \sum_{k=2}^{\infty} g_k \phi^k. \quad (2)$$

The mass of the field is chosen to be 1, and the derivative of the potential  $U(\phi)$  is expanded as a power series in  $\phi$ , where the  $g_k$  are constants. In the following unless otherwise stated we shall consider potentials which are symmetric around their minima, i.e.  $g_{2k} = 0$ .

The small-amplitude expansion of solutions of Eq. (2)

can be written as

$$\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_k, \quad (3)$$

where  $\varepsilon \ll 1$  is the expansion parameter. Demanding the functions  $\phi_k$  be bounded they all turn out to be periodic in time. For the class of symmetric potentials  $\phi_{2k} = 0$ ,  $k = 1, 2, \dots$ , and the first two non-vanishing amplitudes are given as

$$\phi_1 = p_1 \cos(\omega t), \quad (4)$$

$$\phi_3 = p_3 \cos(\omega t) - \frac{\lambda}{24} p_1^3 \cos(3\omega t), \quad (5)$$

where  $\lambda = -3g_3/4$ , the frequency,  $\omega = \sqrt{1 - \varepsilon^2}$ , and the functions  $p_1$  and  $p_3$  depend only on the radial coordinate  $r$ . Introducing  $S$  and  $Z$  by

$$p_1 = \frac{S}{\sqrt{\lambda}}, \quad p_3 = \mu Z, \quad \mu = \frac{1}{\lambda^2 \sqrt{\lambda}} \left( \frac{1}{24} \lambda^2 + \frac{5}{8} g_5 \right), \quad (6)$$

the constants  $g_k$  get eliminated from the equations determining  $p_1$  and  $p_3$ ,

$$\frac{d^2 S}{d\rho^2} + \frac{D-1}{\rho} \frac{dS}{d\rho} - S + S^3 = 0, \quad (7)$$

$$\frac{d^2 Z}{d\rho^2} + \frac{D-1}{\rho} \frac{dZ}{d\rho} - Z + 3S^2 Z - S^5 = 0, \quad (8)$$

where a rescaled radial coordinate has been introduced through  $\rho = \varepsilon r$ . Eq. (7) admits regular solutions decaying exponentially in dimensions  $D < 4$ , implying similar properties for the regular solutions of Eq. (8). From now on we will only discuss the fundamental solution of Eq. (7) without nodes. On solutions with nodes see Ref. [37]. The amplitudes  $\phi_{2k+1}$ ,  $k \geq 2$ , are determined by linear inhomogeneous equations analogous to (8), and they all decay exponentially. The small amplitude series (3) does not converge in general – it is an *asymptotic* one. This is closely related to the fact that no true breathers exist in generic theories because of the radiative degrees of freedom. Nevertheless the asymptotic series (3) yields an excellent approximation for the spatially localized “core” part of the oscillon (for details see Ref. [37]).

It turns out that the radiative amplitude of an oscillon is of the order of  $\exp(-B_D/\varepsilon)$  i.e. beyond reach of finite order perturbation theory in  $\varepsilon$ . As found in Ref. [12] any slowly radiating oscillon solution can be well approximated by a suitable exactly time periodic, “quasibreather” (QB) having a well-localized core and an inevitable standing wave tail, whose amplitude is in some sense minimal. In Section 4. the amplitude of this standing wave tail and that of the outgoing radiation will be directly related. To find this amplitude using the Segur-Kruskal method [35], we have to analytically continue our functions to the complex plane and find the singularities closest to the real axis as a first step. Because (8) is linear in  $Z$  its solution has a singularity exactly where  $S$  has. Thus, we have to

find the singularities of the master equation (7). In the neighbourhood of the closest singularity to the real axis we compute the nonperturbative correction, which determines the radiation field of the oscillon after matching it to the solution on the real axis.  $S$  has simple pole singularities on the imaginary axis at  $\rho = \pm i P_D$ . Let us measure the distances from the upper singularity as

$$\rho = i P_D + R. \quad (9)$$

Then the solution of (7) for small  $R$  can be written in the following series form

$$S = -\frac{i\sqrt{2}}{R} + \frac{\sqrt{2}(D-1)}{6P_D} + \mathcal{O}(R). \quad (10)$$

We note that for  $D > 1$  the series (10) will also contain non-analytic terms, the lowest order being  $R^3 \ln R$ . The position of the pole can be determined both by Padé’s approximation and by integrating the master equation numerically along the imaginary axis. Table 1 gives the results of these numerical approaches. For  $D = 4$   $P_D = 0$ ,

$D$	$P_D$
1	1.57080
2	1.09256
3	0.60218

Table 1: The distance between the real axis and the pole of the fundamental solution of the master equation (7) as a function of the number of spatial dimensions  $D$ .

i.e. the pole is in the origin. It has been shown in Ref. [37] that there are no regular and localized solutions of Eq. (7) for  $D \geq 4$ . This implies that in dimensions  $D \geq 4$  the oscillon core is not localized. As it will be shown the pole term  $P_D$  appears in the exponential in the radiation law, and from Table 1 it follows that oscillons in higher dimensions will radiate faster.

The solution of the first inhomogeneous equation (8) in the neighborhood of the singularity takes the form

$$Z = -\frac{i2\sqrt{2}}{3} \frac{1}{R^3} - \frac{8\sqrt{2}(D-1)}{15P_D} \frac{\ln R}{R^2} + \frac{z_{-2}}{R^2} + \dots \quad (11)$$

The non-analytic terms represent dimensional corrections to the one-dimensional results. The free constant  $z_{-2}$  is due to the solution of the homogeneous part of the equation for  $Z$  and is determined by the matching conditions, i.e. by using the fact that  $Z$  is decaying on the real axis.

We have seen that the small-amplitude expansion yields time periodic and spatially localized (of finite energy) configurations which appear to be breathers. On the other hand any time periodic solution of Eq. (2) can be expanded into a Fourier series as:

$$\phi = \sum_{n=0}^{\infty} \cos(n\omega t) \Phi_n. \quad (12)$$

$\Phi_{2n} = 0$  in the case of symmetric potentials presently considered. For later use we compare the two different expansions, (3) and (12):

$$\Phi_1 = \varepsilon p_1 + \varepsilon^3 p_3 + \mathcal{O}(\varepsilon^5), \quad \Phi_3 = -\frac{\lambda}{24} \varepsilon^3 p_1^3 + \mathcal{O}(\varepsilon^5). \quad (13)$$

Next we determine the behaviour of the Fourier modes,  $\Phi_n$ , near the singularity. We define a new spatial coordinate  $y$  by  $R = \varepsilon y$ , which is related to  $r$  as:

$$r = \frac{i P_D}{\varepsilon} + y. \quad (14)$$

The “inner region”  $R \ll 1$  is not small in the  $y$  coordinates; if  $\varepsilon \rightarrow 0$  then  $\varepsilon|y| = |R| \ll 1$  but  $|y| \rightarrow \infty$ . Then from Eqs. (10) and (11) the asymptotic behaviour for  $|y| \rightarrow \infty$  of  $S$  and  $Z$  can be rewritten as

$$\varepsilon S = \frac{-i\sqrt{2}}{y} + \varepsilon \frac{\sqrt{2}(D-1)}{6P_D} + \dots \quad (15)$$

$$\begin{aligned} \varepsilon^3 Z = & -\frac{i2\sqrt{2}}{3} \frac{1}{y^3} - \varepsilon \ln \varepsilon \frac{8\sqrt{2}(D-1)}{15P_D} \frac{1}{y^2} \\ & - \varepsilon \frac{8\sqrt{2}(D-1)}{15P_D} \frac{\ln y}{y^2} + \varepsilon \frac{z_{-2}}{y^2} + \dots, \end{aligned} \quad (16)$$

and from Eqs. (13), (6) we obtain

$$\begin{aligned} \Phi_1 = & -\frac{i\sqrt{2}}{\sqrt{\lambda}} \frac{1}{y} - \frac{2i\mu\sqrt{2}}{3} \frac{1}{y^3} - \varepsilon \ln \varepsilon \frac{8\mu\sqrt{2}(D-1)}{15P_D} \frac{1}{y^2} \\ & + \varepsilon \left[ \frac{\sqrt{2}(D-1)}{6\sqrt{\lambda}P_D} + \mu \left( \frac{z_{-2}}{y^2} - \frac{8\sqrt{2}(D-1)}{15P_D} \frac{\ln y}{y^2} \right) \right] + \dots, \end{aligned} \quad (17)$$

$$\Phi_3 = -\frac{i\sqrt{2}}{12\sqrt{\lambda}} \frac{1}{y^3} + \varepsilon \frac{\sqrt{2}(D-1)}{24\sqrt{\lambda}P_D} \frac{1}{y^2} + \dots \quad (18)$$

We shall impose Eqs. (17), (18) as boundary conditions to construct the inner solution (close to the singularity). We note that there is also a term proportional to  $\varepsilon \ln \varepsilon / y^4$  in  $\Phi_3$  arising from  $\varepsilon^5$  order terms in the small-amplitude expansion.

To leading-order, i.e. setting  $\varepsilon = 0$  in (17) and (18), we recover the one-dimensional result obtained in [34]. The first dimensional correction comes from the  $\varepsilon \ln \varepsilon$  terms. As we shall see, this will give a multiplicative correction factor to the energy loss rate. In the case of the SG model for  $D = 1$  the breathers do not radiate. To determine the radiation loss in the SG models for  $D > 1$  we have to calculate the next order corrections, arising from the  $\varepsilon$  terms.

### 3. Fourier mode expansion near the pole

Let us now directly Fourier decompose our field equation (2) using (12), without employing the small-amplitude expansion,

$$\left[ \frac{d^2}{dr^2} + \frac{D-1}{r} \frac{d}{dr} + (n^2 \omega^2 - 1) \right] \Phi_n = F_n, \quad (19)$$

where the nonlinear terms on the right hand side have the form

$$F_n = \frac{g_3}{4} \sum_{m,p,q=1}^{\infty} \Phi_m \Phi_p \Phi_q \delta_{n, \pm m \pm p \pm q} + \dots \quad (20)$$

Then in the inner region the mode equations (19) keeping also  $\mathcal{O}(\varepsilon)$  correction can be written as

$$\left[ \frac{d^2}{dy^2} + \varepsilon \frac{D-1}{iP_D} \frac{d}{dy} + (n^2 - 1) + \mathcal{O}(\varepsilon^2) \right] \Phi_n = F_n. \quad (21)$$

By expanding equations (21) directly in powers of  $1/y$ , including  $\ln y$  terms when necessary one can obtain Eqs. (17), (18). Technically it is easier to obtain (17) and (18) this way then by the small  $\varepsilon$  expansion.

Imposing Eqs. (17), (18) (and the corresponding ones for  $n > 3$ ) as boundary conditions for  $\text{Re } y \rightarrow \infty$  on the Fourier modes  $\Phi_n$  defines a unique solution of the system (21). This corresponds to the behavior of the (real) solution for which all  $\Phi_n$  decay exponentially when  $r \rightarrow \infty$  on the real axis. Since in general no true breather exists, this decaying solution is singular at  $r = 0$ . Its extension to the complex plane is not real on the  $\text{Re } y = 0$  axis, the imaginary parts of the modes  $\Phi_n$  satisfy homogeneous equations to leading-order in  $1/y$ . Specifically, the solution for  $\Phi_3$  behaves as

$$\text{Im } \Phi_3 = \nu_3 \exp(-i\sqrt{8}y) \quad \text{for } \text{Re } y = 0. \quad (22)$$

The constant  $\nu_3$  determines the leading-order part of the radiation amplitude in the  $\Phi_3$  mode which we aim to compute. A method to find the value of  $\nu_3$  developed in Ref. [35] is to integrate Eqs. (21) along a constant  $\text{Im } y$  line numerically, starting from the values given by (17), (18) for large  $|y|$ . An analytic method, using Borel summation, has been proposed in Refs. [38, 39] and has been adapted for one-dimensional scalar oscillons in [34].

Since to zeroth order in  $\varepsilon$  Eq. (21) is the same as for  $D = 1$ , the constant  $\nu_3$  is independent of  $D$  for  $\varepsilon \rightarrow 0$ , corresponding to near threshold states with  $\omega \rightarrow 1$ . In order to find the corrections of order  $\mathcal{O}(\varepsilon \ln \varepsilon)$  and of  $\mathcal{O}(\varepsilon)$  to the value of  $\nu_3$ , it is sufficient to solve the mode equations linearized about the one-dimensional solution. Denoting the solution of the equations for  $D = 1$  by  $\Phi_n^{D=1} + \mathcal{O}(\varepsilon^2)$ , and writing  $\Phi_n = \Phi_n^{D=1} + \tilde{\Phi}_n$ , the equations linearized in  $\varepsilon$  around  $D = 1$  take the form

$$\begin{aligned} & \left[ \frac{d^2}{dy^2} + (n^2 - 1) \right] \tilde{\Phi}_n + \varepsilon \frac{D-1}{iP_D} \frac{d}{dy} \Phi_n^{D=1} \\ & = \sum_{m=\text{odd}} \frac{\partial F_n}{\partial \Phi_m} \Big|_{\Phi_n = \Phi_n^{D=1}} \tilde{\Phi}_m. \end{aligned} \quad (23)$$

$\tilde{\Phi}_n$  contains in general corrections of order  $\mathcal{O}(\varepsilon \ln \varepsilon)$  and of  $\mathcal{O}(\varepsilon)$ . The solution of the linearized equations to order  $\mathcal{O}(\varepsilon \ln \varepsilon)$  can be explicitly written as

$$\tilde{\Phi}_n = \varepsilon \ln \varepsilon C \frac{d}{dy} \Phi_n^{D=1}, \quad (24)$$

where  $C$  is an arbitrary constant independent of  $n$ . Eq. (24) corresponds simply the zero mode when  $D = 1$  in Eq. (23). The constant  $C$  is determined by the matching conditions (17) and (18):

$$C = i\sqrt{\lambda\mu} \frac{8(D-1)}{15P_D}. \quad (25)$$

In order to apply the Borel summation method described in detail in [34], one expands the mode equations near the singularity, e.g. for  $\Phi_3^{D=1}$ :

$$\Phi_3^{D=1} = i \sum_{k=2}^{\infty} \frac{B_k}{y^{2k-1}}, \quad (26)$$

and determines the growth of the coefficients  $B_k$  for large  $k$  as a first step. As found in Ref. [34]:

$$B_k \sim K^{D=1} (-1)^k \frac{(2k-2)!}{8^{k-1/2}}. \quad (27)$$

The Borel sum of Eq. (26) yields the imaginary part (in the case of symmetric potentials) of  $\Phi_3^{D=1}$  and by comparing it to Eq. (22) we have found the connection between the constant  $K$  and  $\nu_3$ , yielding  $\nu_3^{D=1} = K^{D=1} \pi/2$ . Expanding the mode equation near the singularity and calculating  $K$  from the behaviour of  $B_k$  for large  $k$  is a much easier and more precise way to calculate  $\nu_3$  than numerical integration. Since the leading-order dimensional corrections, of order  $\varepsilon \ln \varepsilon$  are given by (24), the dominant dimensional contribution to the value of  $K$  is given as

$$K = K^{D=1} \left(1 - \varepsilon \ln \varepsilon i \sqrt{8C}\right), \quad (28)$$

where  $C$  is given in (25). The relation  $\nu_3 = K\pi/2$  still remains true. In general, this expression already gives satisfactory results for the energy loss of oscillons, except for the SG theory, where  $K^{D=1} = 0$ . Then the correction of order  $\mathcal{O}(\varepsilon)$  become essential.

We now outline the computations of the dimensional corrections of  $\mathcal{O}(\varepsilon)$  to the SG theory, in which case this correction is the leading one to the radiation amplitude. When  $K^{D=1} \neq 0$  there are other contributions to this order besides the dimensional correction, such as  $1/y$  corrections to Eq. (22) (see Eq. (23) of [34]) and the interaction of the outgoing wave with the core and these we do not compute here. It has already been pointed out that to each term of order  $\mathcal{O}(\varepsilon \ln \varepsilon)$  there corresponds a term of order  $\mathcal{O}(\varepsilon)$ , which is obtained by changing  $\ln \varepsilon$  to  $\ln y$ . Hence introducing  $\bar{\Phi}_n$  by

$$\tilde{\Phi}_n = \varepsilon \ln \varepsilon C \frac{d}{dy} \Phi_n^{D=1} + \varepsilon \left( C \ln y \frac{d}{dy} \Phi_n^{D=1} + \bar{\Phi}_n \right), \quad (29)$$

the linearized equation (23) takes the form

$$\left[ \frac{d^2}{dy^2} + (n^2 - 1) \right] \bar{\Phi}_n + C \left( \frac{2}{y} \frac{d^2}{dy^2} \Phi_n^{D=1} - \frac{1}{y^2} \frac{d}{dy} \Phi_n^{D=1} \right) + \frac{D-1}{iP_D} \frac{d}{dy} \Phi_n^{D=1} = \sum_{m=\text{odd}} \frac{\partial F_n}{\partial \Phi_m} \Big|_{\Phi_m=\Phi_m^{D=1}} \bar{\Phi}_m. \quad (30)$$

The advantage of using  $\bar{\Phi}_n$  is that there are no  $\ln y$  terms in its  $1/y$  expansion. Expanding now  $\bar{\Phi}_3$  as

$$\bar{\Phi}_3 = \sum_{k=1}^{\infty} \frac{\beta_k}{y^{2k}}, \quad (31)$$

from (30) it follows that to leading-order

$$(2k-2)(2k-1)\beta_{k-1} + 8\beta_k - (2k-1) \frac{D-1}{P_D} \beta_k = 0. \quad (32)$$

To find the large  $k$  behaviour of  $\beta_k$ , we consider the following Ansatz:

$$\beta_k \sim L (-1)^k \frac{(2k)!}{8^k} + M (-1)^k \frac{(2k-1)!}{8^k}, \quad (33)$$

where  $L$  and  $M$  are the constants to be determined. Eq. (32) yields  $L$ ,

$$L = \frac{\sqrt{2}(D-1)}{8P_D} K^{D=1} \quad (34)$$

while  $M$  is still arbitrary to this order in  $k$ . For the SG model  $K^{D=1} = 0$ , and by substituting the  $1/y$  expansion into Eq. (30) we obtain a higher-order asymptotic formula for the behaviour of  $\beta_k$  for large values of  $k$ ,

$$\beta_k \sim M (-1)^k \frac{(2k-1)!}{8^k} \left( 1 + \frac{1}{k} + \frac{3}{4k^2} \right). \quad (35)$$

The value of  $M$  can then be obtained to a good precision by explicitly calculating the coefficients  $\beta_k$  in the expansion of (30) to moderately high orders. Although we always work with a truncated SG potential and with a finite number of modes the value of  $K^{D=1}$  can be made very small by using  $\Phi_1 - \Phi_9$  and truncating the potential at  $12^{\text{th}}$  order. For the constant  $K$  of the SG theory we obtain

$$K = \varepsilon M, \quad M = 0.6011 \frac{D-1}{P_D}. \quad (36)$$

We note that although a term proportional to  $z_{-2}$  (defined in (11)) also appears in the obtained values of  $\beta_k$ , its coefficient quickly tends to zero when increasing the order of truncation. Repeating the Borel summation argument for this case, it turns out that  $\nu_3 = K\pi/2$  still holds, giving the behaviour of  $\text{Im } \Phi_3$  on the imaginary axis in the neighbourhood of the singularity. Naturally, the next step is to continue the imaginary correction back to the real axis.

#### 4. Continuation to the real axis

The standing wave tail of a small amplitude time periodic QB satisfies to leading order a homogeneous linear equation given by the left hand side of (19). Since the size of a QB grows proportionally to  $1/\varepsilon$ , it is possible to consider an inner region, which is, however, outside of the domain around  $r = 0$  where the asymptotically decaying modes,  $\Phi_n$ , get large. In the previous section we constructed the function  $\delta\Phi_3$  by Borel-summing the

asymptotic series. We extend  $\delta\Phi_3$  to the inner region assuming that it behaves as (22) close to the upper pole, and as  $\text{Im}\Phi_3 = -\nu_3 \exp(i\sqrt{8}y)$  near the lower pole, where  $r = -iP_D/\varepsilon + y$ . The resulting function for large  $r$  is

$$\delta\Phi_3 = i\nu_3 \left(\frac{P_D}{\varepsilon r}\right)^{(D-1)/2} \exp\left(-\frac{\sqrt{8}P_D}{\varepsilon}\right) \times \left[i^{(D-1)/2} \exp(-i\sqrt{8}r) - (-i)^{(D-1)/2} \exp(i\sqrt{8}r)\right]. \quad (37)$$

The general solution of the left hand side of (19) can be written as a sum involving Bessel functions  $J_n$  and  $Y_n$ , which have the asymptotic behaviour

$$J_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \quad (38)$$

$$Y_\nu(x) \rightarrow \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad (39)$$

for  $x \rightarrow +\infty$ . The solution satisfying the asymptotics given by (37) is

$$\delta\Phi_3 = \sqrt[4]{2}\sqrt{\pi} \frac{\alpha_D}{r^{D/2-1}} Y_{D/2-1}(\sqrt{8}r), \quad (40)$$

where the amplitude at large  $r$  is given by

$$\alpha_D = 2\nu_3 \left(\frac{P_D}{\varepsilon r}\right)^{(D-1)/2} \exp\left(-\frac{\sqrt{8}P_D}{\varepsilon}\right). \quad (41)$$

This is just the solution singular at  $r = 0$ . The singularity is the consequence of the initial assumption of exponential decay for large  $r$ . The asymptotic decay induces an oscillation given by (40) in the intermediate core, and a singularity at the center. In contrast, the QB solution has a regular center, but contains a minimal amplitude standing wave tail asymptotically. Considering the left hand side of (19) as an equation describing perturbation around the asymptotically decaying solution, we just have to add a solution  $\delta\Phi_3$  determined by the amplitude (41) with the opposite sign of (40) to cancel the oscillation and the singularity in the core. This way one obtains the regular QB solution, whose minimal amplitude standing wave tail is given as

$$\begin{aligned} \phi_{QB} &= -\sqrt[4]{2}\sqrt{\pi} \frac{\alpha_D}{r^{D/2-1}} Y_{D/2-1}(\sqrt{8}r) \cos(3t) \\ &\approx -\frac{\alpha_D}{r^{(D-1)/2}} \sin\left[\sqrt{8}r - (D-1)\frac{\pi}{4}\right] \cos(3t). \end{aligned} \quad (42)$$

Adding the regular solution, where  $Y$  is replaced by  $J$ , would necessarily increase the asymptotic amplitude.

Subtracting the regular solution with a phase shift in time, we cancel the incoming radiating component, and obtain the radiative tail of the oscillon,

$$\phi_{osc} = -\sqrt[4]{2}\sqrt{\pi} \frac{\alpha_D}{r^{D/2-1}} \quad (43)$$

$$\begin{aligned} &\times \left[Y_{D/2-1}(\sqrt{8}r) \cos(3t) - J_{D/2-1}(\sqrt{8}r) \sin(3t)\right] \\ &\approx -\frac{\alpha_D}{r^{(D-1)/2}} \sin\left[\sqrt{8}r - (D-1)\frac{\pi}{4} - 3t\right]. \end{aligned} \quad (44)$$

The radiation law of the oscillon is easily obtained now,

$$\frac{dE}{dt} = -3\sqrt{2}\pi^2 \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} K^2 \left(\frac{P_D}{\varepsilon}\right)^{D-1} \exp\left(-\frac{2\sqrt{8}P_D}{\varepsilon}\right). \quad (45)$$

Without going into details in the case of asymmetric potentials, we notice that once the one-dimensional problem is solved the first dimensional correction of order  $\varepsilon \ln \varepsilon$  is obtained in essentially the same manner,

$$\begin{aligned} K &= K^{D=1} \left(1 - \varepsilon \ln \varepsilon i\sqrt{3}C\right), \quad \lambda = \frac{5}{6}g_2^2 - \frac{3}{4}g_3, \quad (46) \\ C &= \frac{i}{\lambda^2} \left(\frac{\lambda^2}{24} - \frac{\lambda}{6}g_2^2 + \frac{5}{8}g_5 - \frac{7}{4}g_2g_4 + \frac{35}{27}g_2^2\right) \frac{8(D-1)}{15P_D}. \end{aligned}$$

The radiation law for small-amplitude oscillons is then

$$\frac{dE}{dt} = -\sqrt{3}\pi^2 \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2})} K^2 \left(\frac{P_D}{\varepsilon}\right)^{D-1} \exp\left(-\frac{2\sqrt{3}P_D}{\varepsilon}\right). \quad (47)$$

Eqs. (45) and (47) can be interpreted as an evolution equation for oscillons of the corresponding theory. As the radiation power is exponentially small, the oscillons evolve through undistorted quasibreather states adiabatically. The oscillon energy was calculated in Ref. [37] and is  $E = \varepsilon^{2-D} \frac{E_0}{2\lambda} + \varepsilon^{4-D} E_1$  as a function of  $\varepsilon$ . Plugging this expression into the radiation law gives us the evolution equation of  $\varepsilon$ .

## 5. Numerical simulations

In this section we will provide numerical results regarding oscillon radiation. We have investigated a symmetric  $\phi^6$  theory in  $D = 2$ , and the sine-Gordon theory in  $D = 2$  and  $D = 3$ . The results of numerical simulations in these theories confirm the theoretical predictions with satisfactory accuracy. The time evolution of oscillons was simulated using the fourth order method of line code developed in Refs. [40, 41]. Initial data was obtained by applying the small-amplitude expansion method to  $\varepsilon^3$  order. A one parameter fine tuning, by multiplying the amplitude with a factor close to one, was applied. For  $D = 2$  this minimized the low frequency modulation of the oscillon state, while in  $D = 3$  the tuning was used to suppress the single decaying mode.

### 5.1. Oscillons in the symmetric $\phi^6$ theory in $D = 2$

In Ref. [34] we have found that oscillons of the symmetric  $\phi^6$  theory defined by the potential

$$U(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{1}{6}\phi^6, \quad (48)$$

obeyed the theoretical radiation law in  $D = 1$  to a satisfactory precision. In this theory the radiation of small-amplitude oscillons is maximal among symmetric  $\phi^6$  theories, therefore we have a chance of observing the dimensional correction to the value of  $K$ . The  $\varepsilon$  dependence of



$\epsilon$	$W$
0.33398	$-3.4177 \cdot 10^{-8}$
0.33113	$-1.5053 \cdot 10^{-7}$
0.31125	$-1.6574 \cdot 10^{-7}$
0.29829	$-8.9890 \cdot 10^{-8}$
0.29098	$-2.7064 \cdot 10^{-8}$
0.28691	$-1.9475 \cdot 10^{-8}$
0.28017	$-8.3725 \cdot 10^{-9}$
0.27030	$-1.5860 \cdot 10^{-11}$
0.25794	$-3.4518 \cdot 10^{-10}$
0.24186	$-4.0292 \cdot 10^{-10}$
0.22925	$-1.5740 \cdot 10^{-10}$
0.22375	$-9.8124 \cdot 10^{-11}$
0.21291	$-3.2468 \cdot 10^{-11}$
0.20222	$-9.0850 \cdot 10^{-12}$

Table 2: Radiation power  $W$  for the oscillons in the  $\phi^6$  theory.

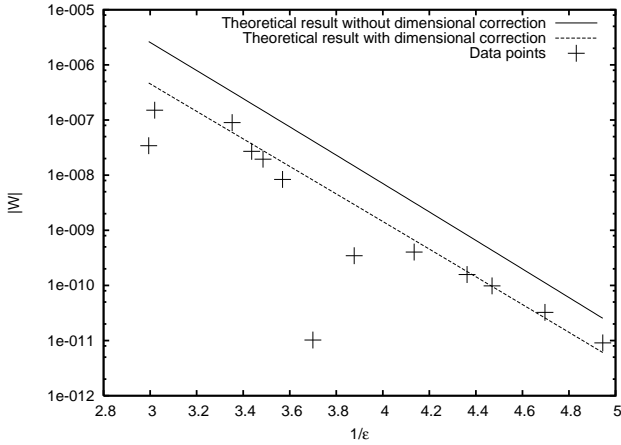


Figure 1: Radiation law for small-amplitude oscillons in the  $\phi^6$  theory for  $D = 2$ . We plotted the theoretical radiation law with the first dimensional correction and without it.

the energy loss rate  $W = dE/dt$  is listed in Table 2 and plotted on Fig. 1. Analogously to the  $D = 1$  case, we find anomalous points, corresponding to oscillons which radiate much slower than the theoretical prediction. This reveals the rich structure of finite  $\epsilon$  corrections and outlines the limitations of the calculation in the small-amplitude limit.

Based on the theoretical radiation law (45), in the  $D = 2$  case we use a semiempirical formula

$$\frac{dE}{dt} = -K^2 \frac{287.45}{\epsilon} \exp\left(-\frac{6.18045 b}{\epsilon}\right), \quad (49)$$

where an empirical correction constant  $b$  is included in the exponent, and the dimensional correction is included in the form of (28), with  $i\sqrt{8}C = -1.592$  in this case. The theoretical value of  $K^{D=1}$  has been obtained in [34],  $K^{D=1} = 0.579$ , while the theoretical value of  $b$  is obviously  $b = 1$ . We performed fits for the points with smaller  $\epsilon$  values than the last anomalous one. Fitting

for both  $K^{D=1}$  and  $b$  for  $\epsilon < 0.25$ , we have obtained  $K^{D=1} = 0.05 \pm 0.02$  and  $b = 0.82 \pm 0.02$ . We get the theoretical pole term in the exponent quite precisely, but the amplitude  $K^{D=1}$  has large error. However, setting  $b = 1$  and fitting only for  $K^{D=1}$  in the same domain, we have obtained  $K^{D=1} = 0.59 \pm 0.05$ , which is close to the theoretical prediction. Interestingly, there are points outside the domain of our fits which also obey the theoretical radiation law. It is evident from Fig. 1 that the dimensional correction significantly decreases the radiation power, but has little effect on the pole term, i.e. the curves have approximately the same slope. The data points back the theoretical form of the dimensional correction, even though the precise  $\epsilon \ln \epsilon$  functional dependence is impossible to verify.

## 5.2. Comments on the $\phi^4$ theory in $D = 2$

Oscillons in the  $\phi^4$  theory show peculiar behavior in  $D = 2$ . In the  $\epsilon$  domain accessible to our lattice simulations they obey the semiempirical radiation law, however it is not consistent with the expected  $\exp(-2\sqrt{3}P_2/\epsilon)$  pole term, even though the outgoing radiation is dominantly in the  $\Phi_2$  mode. We remark that in the  $\phi^4$  case the first dimensional correction is so large in the accessible  $\epsilon$  domain that it invalidates the computed leading-order behavior.

## 5.3. Oscillons in the $D = 2$ and $D = 3$ sine-Gordon theory

In the  $D = 2$  and  $D = 3$  SG theory we expect oscillons to live exceptionally long, as the potential is symmetric and the dimension independent part of  $K$  is zero. Our simulation results are collected in Table 3, and plotted on Figs. 2 and 3.

D=2		D=3	
$\epsilon$	$W$	$\epsilon$	$W$
0.39815	$-1.8395 \cdot 10^{-5}$	0.32513	$5.9630 \cdot 10^{-3}$
0.35141	$-1.6531 \cdot 10^{-6}$	0.30801	$3.2942 \cdot 10^{-3}$
0.33637	$-6.8685 \cdot 10^{-7}$	0.28520	$1.2931 \cdot 10^{-3}$
0.30090	$-6.0986 \cdot 10^{-8}$	0.26350	$4.6372 \cdot 10^{-4}$
0.27560	$-7.5049 \cdot 10^{-9}$	0.24218	$1.4336 \cdot 10^{-4}$
0.25037	$-5.3072 \cdot 10^{-10}$	0.22130	$3.7224 \cdot 10^{-5}$
		0.20076	$7.6601 \cdot 10^{-6}$
		0.18042	$1.1435 \cdot 10^{-6}$

Table 3: Radiation power in the SG theory.

For the  $D = 2$  case, we can again use the semiempirical formula (49), where now  $K = \epsilon M$ . According to (36), the theoretical value of  $M$  is  $M = 0.5502$ , and for the other parameter, obviously  $b = 1$ . First we fitted both  $M$  and  $b$ , obtaining  $M = 1.79 \pm 0.16$  and  $b = 1.08 \pm 0.01$ . Fitting only  $M$  while fixing  $b = 1$  we got  $M = 0.76 \pm 0.04$ . Similarly to the  $\phi^6$  theory, the pole term can be measured quite precisely, but the amplitude has larger errors.

Motivated by (45), in the  $D = 3$  case we use the semiempirical formula

$$\frac{dE}{dt} = -K^2 \frac{190.81}{\epsilon^2} \exp\left(-\frac{3.40644 b}{\epsilon}\right), \quad (50)$$

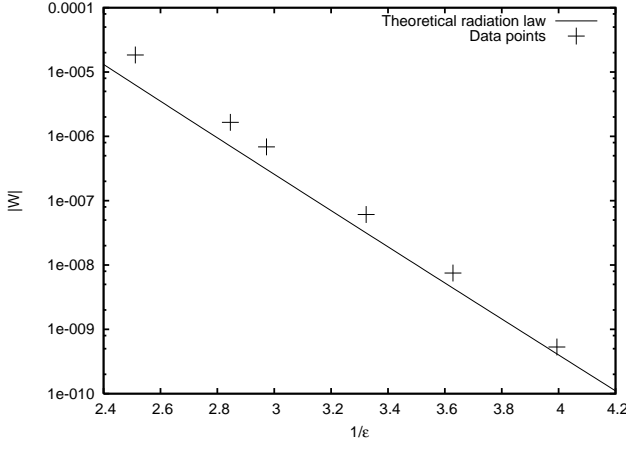


Figure 2: The radiation law for oscillons in the  $D = 2$  SG theory.

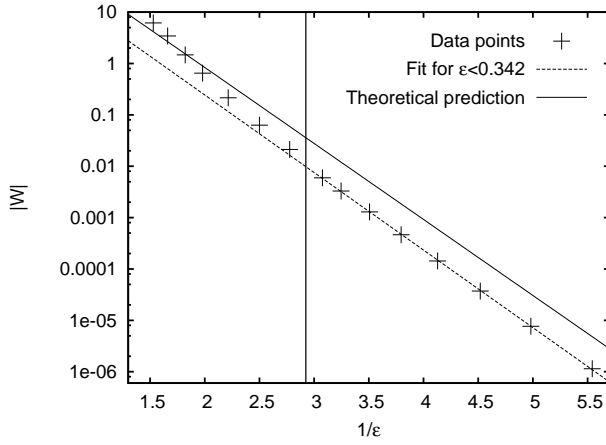


Figure 3: The radiation law for oscillons in the  $D = 3$  SG theory. The vertical line corresponds to the energy minimum at  $\varepsilon = 0.342$ . Points to the left of the line correspond to stable oscillon states, while the others to states having one decay mode, requiring a one-parameter tuning.

where  $K = \varepsilon M$ . The theoretical values are  $M = 1.9964$  and  $b = 1$ . The fit for  $\varepsilon < 0.342$  yields  $M = 1.156 \pm 0.04$  and  $b = 1.02047 \pm 0.004$ . Again, the pole term is precise, but the amplitude has larger error. Setting  $b = 1$  and fitting only for  $M$  does not help in this case, it gives about half of the theoretical value,  $M = 1.0019 \pm 0.01$ . Most probably this discrepancy is due the nonlinear effects related to the still too large values of  $\varepsilon$ . Since the energy loss rate falls exponentially with  $\varepsilon$ , our time evolution code cannot provide reliable results for smaller  $\varepsilon$  values.

## 6. Acknowledgments

This research has been supported by OTKA Grants No. K61636, NI68228.

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